BMath II - Semestral Examination - Topology

Time : 3 Hours Max. Marks : 60

Answer all questions. Give complete justifications.

(1) Let $A \subseteq \mathbb{R}^2$ be a countable subset. Show that $\mathbb{R}^2 - A$ is path connected. [12]

(2) Construct a homeomorphism
$$f : \{(x, y) \in \mathbb{R}^2 \ : \ y \geq 0\} \longrightarrow \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1, (x, y) \neq (0, 1)\}.$$
Both the spaces have the subspace topology of $\mathbb{R}^2$. [12]

(3) Let $A \subseteq \mathbb{R}^\omega$ be the subset
$$A = \{(x_i) \in \mathbb{R}^\omega \ : \ x_i = 0 \text{ for all but finitely many } i\}.$$
Prove that A is dense in $\mathbb{R}^\omega$ with the product topology. Is A dense in $\mathbb{R}^\omega$ with the box topology? Recall that, as a set, $\mathbb{R}^\omega$ is the countable product of $\mathbb{R}$ with itself. [6+6]

(4) Let $X = \mathbb{R} \times \{0, 1\}$ with the product topology where $\{0, 1\}$ has the discrete topology. Let $\sim$ be the equivalence relation on X defined as follows. Given $x, x' \in \mathbb{R}$ and $t, s \in \{0, 1\}$ we declare
$$(x, t) \sim (x', s)$$
if and only if one of the following is satisfied
(a) $x = x'$, $|x| > 1$
(b) $x = x'$, $|x| \leq 1$ and $t = s$.
Let X^* denote the set of equivalence classes of X under the above equivalence relation. We give X^* the quotient topology with respect to the surjective function $p : X \longrightarrow X^*$ which takes $y \in X$ to the equivalence class $[y]$ of y. Decide whether X^* with the quotient topology satisfies the T_1 and Hausdorff axioms. (You may want to draw diagrams) [6+6]

(5) Given $x, y \in \mathbb{R}$ define
$$d(x, y) = |e^{-x} - e^{-y}|.$$
Show that d is a metric on $\mathbb{R}$. Decide whether $(\mathbb{R}, d)$ is a complete metric space. Does d induce the usual topology on $\mathbb{R}$? [2+8+2]